On noncommutative sinh-Gordon equation

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Abstract

We give a noncommutative extension of sinh-Gordon equation. We generalize a linear system and Lax representation of the sinh-Gordon equation in noncommutative space. This generalization gives a noncommutative version of the sinh-Gordon equation with extra constraints, which can be expressed as global conserved currents.

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1 Introduction

Noncommutative geometry has widely been used in the study of integrable field theories (IFTs) since the last decade [1]-[13]. The noncommutative version of integrable field theories (nc-IFTs) is obtained by replacing ordinary product with $\star-$ star product. In the commutative limit, these noncommutative theories reduce to ordinary field theories. Noncommutative version of different integrable models, such as principal chiral model with and without a Wess-Zumino term, sine-Gordon and sinh-Gordon equations, Korteweg de Vries (KdV) equation, Boussinesq equation, Kadomtsev-Petviashvili (KP) equation, Sawada-Kotera equation, nonlinear Schrodinger equation and Burgers equation, have been studied [1]-[13].

The noncommutativity of space is characterized by

$$\left[x^i, x^j\right] = \mathrm{i}\theta^{ij},$$

where θ^{ij} is a constant second rank tensor, called parameter of noncommutativity. The $\star-$ star product of two functions in noncommutative spaces is given by

$$\left(f \star g\right)(x) = f(x)g(x) + \frac{\mathrm{i}\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + \vartheta(\theta^2), \qquad \text{where } \partial_i = \tfrac{\partial}{\partial x^i}.$$

The $\star-$ star product obeys following properties:

$$\begin{aligned} f \star I &= f = I \star f, \\ f \star (g \star h) &= (f \star g) \star h. \end{aligned}$$

The integration of two functions is [3]

$$\int d^D x f(x) \star g(x) = \int d^D x f(x) g(x),$$

where the integration is taken in all noncommutative directions.

The noncommutativity in $(1+1)-$ dimensional space-time is defined as [7]-[9]

$$[t, x] = \mathrm{i}\theta.$$

The $\star-$ star product of two functions in $(1+1)$-dimensional noncommutative space is given by [7]-[9]

$$\left(f \star g\right)(t, x) = f(t,x)g(t,x) + \frac{\mathrm{i}\theta}{2}\left(\partial_{t'}\partial_{x''} - \partial_{t''}\partial_{x'}\right) f(t',x')g(t'',x'')|_{t=t'=t'';\ x=x'=x''} + \vartheta(\theta^2),$$

with $\partial_t = \frac{\partial}{\partial t}$. In what follows, we present a noncommutative version of sinh-Gordon equation and investigate the noncommutative version of its zero-curvature and Lax representations. The integrability condition of the linear system and the Lax equation gives rise to a noncommutative sinh-Gordon equation with some extra constraints.

In section 2, we give noncommutative generalization of a linear system whose compatibility condition is the noncommutative sinh-Gordon equation. In section 3, we present a noncommutative version of Lax representation of the noncommutative sinh-Gordon equation. In Section 4, we expand the fields perturbatively and obtain zeroth and first order sinh-Gordon equations, the associated linear system and a set of parametric Bäcklund transformation (BT) of the sinh-Gordon equation. It has been shown that the compatibility condition of the associated linear system and the Bäcklund transformation (BT) is the sinh-Gordon equation at the perturbative level. Section 5, contains our conclusions.

2 Linear System for Noncommutative sinh-Gordon Equation

In this section we discuss the integrability of noncommutative extension of the sinh-Gordon equation. We start with an associated linear system of the equation different from the one given in Ref.[3] and show that its compatibility condition is the noncommutative sinh-Gordon equation along with some constraints. The constraints obtained here are different from those obtained in Ref.[3]. These constraints are also shown to be expressed as conserved global currents.

In general, a nonlinear evolution equation solvable by inverse scattering method can be expressed as a compatibility condition of a set of linear differential equations. The associated linear system can be related to the isospectral problem and the Lax representation. We now write a linear system whose compatibility condition gives the noncommutative version of the sinh-Gordon equation. The linear system of the sinh-Gordon equation is

$$\partial_{\pm} u = A_{\pm}^{\star} \star u, \tag{2.1}$$

where $A_{\pm}^{\star}$ are

$$\begin{aligned} A_{+}^{\star} &= \begin{pmatrix} -\mathrm{i}\lambda & \frac{\beta}{2}\partial_{+}\varphi \\ \frac{\beta}{2}\partial_{+}\varphi & \mathrm{i}\lambda \end{pmatrix}, \\ A_{-}^{\star} &= \frac{\mathrm{i}m^2}{4\lambda} \begin{pmatrix} \cosh_{\star}\beta\varphi & -\sinh_{\star}\beta\varphi \\ \sinh_{\star}\beta\varphi & -\cosh_{\star}\beta\varphi \end{pmatrix}. \end{aligned}$$

with φ a real valued function and β, m are some positive parameters. The compatibility condition of the linear system (2.1) in noncommutative space is the zero-curvature condition:

$$\left[\partial_{+} - A_{+}^{\star}, \partial_{-} - A_{-}^{\star}\right]_{\star} \equiv \partial_{-}A_{+}^{\star} - \partial_{+}A_{-}^{\star} + \left[A_{+}^{\star}, A_{-}^{\star}\right]_{\star} = 0,$$

where $\left[A_{+}^{\star}, A_{-}^{\star}\right] = A_{+}^{\star} \star A_{-}^{\star} - A_{-}^{\star} \star A_{+}^{\star}$ is a commutator in noncommutative space. The above compatibility condition gives rise to the noncommutative sinh-Gordon equation and some extra constraints

$$\begin{aligned} &\partial_{-}\partial_{+}\varphi = \frac{m^2}{\beta}\sinh_{\star}\beta\varphi, \\ &\partial_{+}\left(\cosh_{\star}\beta\varphi\right) - \frac{\beta}{2}(\sinh_{\star}\beta\varphi \star \partial_{+}\varphi + \sinh_{\star}\beta\varphi \star \partial_{+}\varphi) = 0, \\ &\partial_{+}\left(\sinh_{\star}\beta\varphi\right) - \frac{\beta}{2}(\cosh_{\star}\beta\varphi \star \partial_{+}\varphi + \cosh_{\star}\beta\varphi \star \partial_{+}\varphi) = 0. \end{aligned}$$

These constraints become total derivatives or as global conserved currents. We note that in the limit $\theta \to 0$, the first equation becomes the ordinary sinh-Gordon equation and extra constraints vanish, the $\star-$ product becomes the usual product.

3 Lax Representation for Noncommutative sinh-Gordon Equation

In this section we present the Lax representation of the noncommutative sinh-Gordon equation and find the corresponding Lax equations.

The eigenvalue equation for a Lax operator $L_\pm$ is

$$L_\pm \Psi = \lambda \Psi,$$

where $L_\pm$ is given by

$$L_\pm = \begin{pmatrix} -\mathrm{i}\partial_\pm & \frac{\beta}{2}\partial_\pm\varphi \\ \frac{\beta}{2}\partial_\pm\varphi & \mathrm{i}\partial_\pm \end{pmatrix},$$

and

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

The corresponding Lax equations are

$$\partial_\mp L_\pm = [L_\pm, M]_\star\,, \tag{3.1}$$

with

$$M = \frac{\mathrm{i}m^2}{4\lambda}\begin{pmatrix} \cosh_\star \beta\varphi & -\sinh_\star \beta\varphi \\ \sinh_\star \beta\varphi & -\cosh_\star \beta\varphi \end{pmatrix}.$$

The Lax equation gives noncommutative sinh-Gordon equation along with some constraints

$$\begin{aligned}
&\partial_-\partial_+\varphi = \frac{m^2}{\beta}\sinh_\star \beta\varphi, \\
&\partial_\pm(\cosh_\star \beta\varphi) - \frac{\beta}{2}(\sinh_\star \beta\varphi \star \partial_\pm\varphi + \sinh_\star \beta\varphi \star \partial_\pm\varphi) = 0, \\
&\partial_\pm(\sinh_\star \beta\varphi) - \frac{\beta}{2}(\cosh_\star \beta\varphi \star \partial_\pm\varphi + \cosh_\star \beta\varphi \star \partial_\pm\varphi) = 0.
\end{aligned}$$

These constraints become total derivatives (global currents). The first equation in the limit when the noncommutativity parameter reduces to zero becomes an ordinary sinh-Gordon equation and the constraints of the model vanish.

4 Perturbative Expansion of sinh-Gordon Model

We can expand the field in power series of noncommutative parameter θ. The expansion of the field φ up to first order is

$$\varphi = \varphi^{(0)} + \theta\varphi^{(1)}.$$

With this expansion we can obtain two pairs of the sinh-Gordon equation

$$
\begin{aligned}
\partial_+\partial_-\varphi^{(0)} &= \frac{m^2}{\beta}\sinh\beta\varphi^{(0)}, && (4.1)\\
\partial_+\partial_-\varphi^{(1)} &= m^2\varphi^{(1)}\cosh\beta\varphi^{(0)}. && (4.2)
\end{aligned}
$$

The linear system for equation (4.1) becomes

$$
\partial_\pm u^{(0)} = A_\pm^{(0)} u^{(0)}, \tag{4.3}
$$

where $A_\pm^{(0)}$ are given by

$$
\begin{aligned}
A_+^{(0)} &= \begin{pmatrix} -\mathrm{i}\lambda & \frac{\beta}{2}\partial_+\varphi^{(0)} \\ \frac{\beta}{2}\partial_+\varphi^{(0)} & \mathrm{i}\lambda \end{pmatrix},\\
A_-^{(0)} &= \frac{\mathrm{i}m^2}{4\lambda}\begin{pmatrix} \cosh\beta\varphi^{(0)} & -\sinh\beta\varphi^{(0)} \\ \sinh\beta\varphi^{(0)} & -\cosh\beta\varphi^{(0)} \end{pmatrix}.
\end{aligned}
$$

The compatibility condition for the linear system (4.3) is the zeroth order zero-curvature condition:

$$
\left[\partial_+ - A_+^{(0)}, \partial_- - A_-^{(0)}\right] \equiv \partial_- A_+^{(0)} - \partial_+ A_-^{(0)} + \left[A_+^{(0)}, A_-^{(0)}\right] = 0.
$$

The linear system of the equation for first order reads

$$
\begin{aligned}
\partial_+ u^{(1)} &= A_+^{(0)} u^{(1)} + A_+^{(1)} u^{(0)}, && (4.4)\\
\partial_- u^{(1)} &= A_-^{(0)} u^{(1)} + A_-^{(1)} u^{(0)},
\end{aligned}
$$

where $A_\pm^{(1)}$ are given by

$$
\begin{aligned}
A_+^{(1)} &= \begin{pmatrix} 0 & \frac{\beta}{2}\partial_+\varphi^{(1)} \\ \frac{\beta}{2}\partial_+\varphi^{(1)} & 0 \end{pmatrix},\\
A_-^{(1)} &= \frac{\mathrm{i}m^2}{4\lambda}\begin{pmatrix} \beta\varphi^{(1)}\sinh\beta\varphi^{(0)} & -\beta\varphi^{(1)}\cosh\beta\varphi^{(0)} \\ \beta\varphi^{(1)}\cosh\beta\varphi^{(0)} & -\beta\varphi^{(1)}\sinh\beta\varphi^{(0)} \end{pmatrix}.
\end{aligned}
$$

The compatibility condition for the linear system (4.4) is

$$
\left(\partial_- A_+^{(0)} - \partial_+ A_-^{(0)} + \left[A_+^{(0)}, A_-^{(0)}\right]\right) u^{(1)} + \left(\partial_- A_+^{(1)} - \partial_+ A_-^{(1)} + \left[A_+^{(1)}, A_-^{(0)}\right] + \left[A_+^{(0)}, A_-^{(1)}\right]\right) u^{(0)} = 0.
$$

The Bäcklund transformation for equation (4.1) is

$$
\begin{aligned}
\partial_+\left(\frac{\phi_1^{(0)} - \phi^{(0)}}{2}\right) &= \frac{m\lambda}{\beta}\sinh\beta\left(\frac{\phi_1^{(0)} + \phi^{(0)}}{2}\right), && (4.5)\\
\partial_-\left(\frac{\phi_1^{(0)} + \phi^{(0)}}{2}\right) &= \frac{m}{\beta\lambda}\sinh\beta\left(\frac{\phi_1^{(0)} - \phi^{(0)}}{2}\right),
\end{aligned}
$$

and first order correction to equation (4.5) is

$$\begin{aligned} \partial_+(\frac{\phi_1^{(1)}-\phi^{(1)}}{2}) &= m\lambda(\frac{\phi_1^{(1)}+\phi^{(1)}}{2})\cosh\beta(\frac{\phi_1^{(0)}+\phi^{(0)}}{2}), \\ \partial_-(\frac{\phi_1^{(1)}+\phi^{(1)}}{2}) &= \frac{m}{\lambda}(\frac{\phi_1^{(1)}-\phi^{(1)}}{2})\cosh\beta(\frac{\phi_1^{(0)}-\phi^{(0)}}{2}). \end{aligned} \tag{4.6}$$

The integrability condition of equations (4.5) and (4.5) yields equations (4.1) and (4.2), respectively. To solve equation (4.1) we first reduce the problem to a one-dimensional problem by assuming that solution of equations of motion are independent of time. The first soliton solution of equation (4.1) is

$$\varphi^{(0)} = \frac{4}{\beta}\tanh^{-1}\exp(2mx), \tag{4.7}$$

the corresponding first order correction term to the solution is

$$\varphi^{(1)} = \frac{1}{\sinh(2mx)}.$$

This solves the equation of motion and constraints for the noncommutative sinh-Gordon equation to first order in θ .

5 Conclusions

In summery, we have investigated a noncommutative version of sinh-Gordon equation and discussed some of its properties as an integrable equation. This noncommutative sinh-Gordon equation reduces to an ordinary sinh-Gordon equation and constraints of the model vanish in the commutative limit. The noncommutative version of the linear system (or equivalently zero-curvature representation) and Lax representation give an integrable noncommutative sinh-Gordon equation. The constraints of the model appear as total derivatives. We have also analyzed the integrability of the equation at perturbative level. We have presented a set of Bäcklund transformation for the zeroth order sinh-Gordon equation and the first order correction to the zeroth order Bäcklund transformation. The soliton solution of the equation has been obtained. We have also shown that the $1-$ soliton solution of the noncommutative sinh-Gordon equation solves the equations of motion and its constraints.

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